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AXISYMMETRIC MHD VISCOUS FLOW ABOUT A HIGHLY-SLIPPING SPHERE TRANSLATING PARALLEL WITH A UNIFORM AMBIENT MAGNETIC FIELD

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The slow translation of a highly-slipping sphere with radius a in an unbounded viscous conducting Newtonian liquid with constant viscosity μ and conductivity σ is investigated. The liquid is subject to a steady uniform magnetic field $\mathbf{B}$ parallel to the sphere velocity, flows about the sphere and exerts on its a drag force. The resulting axisymmetric MHD flow is expanded as a serie of fundamental flows earlier gained elsewhere [1, 2] for a different Oseen flow problem. The coefficients entering in the serie are determined by enforcing the impermeability and zero tangent stress conditions on the sphere surface, As a result, the highly-slipping sphere drag coefficient C_d is numerically obtained and its sensitivity to the problem Hartmann number $\mathrm{Ha} = a|\mathbf{B}|/\sqrt{\mu/\sigma}$ is examined. Moreover, a polynomial handy formula for C_d is proposed for $\mathrm{Ha} \leq O(1)$ and the computed velocity patterns are presented and discussed for Ha=1,10.

Introduction.

As sketched in Fig. 1, we consider a viscous axisymmetric MHD flow about a highly-slipping sphere S of radius a and center O, translating in a conducting Newtonian liquid at the velocity $\mathbf{U} = U\mathbf{e}_z$ parallel to a prescribed uniform ambient steady magnetic field $\mathbf{B} = B\mathbf{e}_z$ (with $B > 0$).

This flow has no swirl, pressure p and velocity $\mathbf{u}$ with the typical magnitude $|U| > 0$. In general, both the electric field $\mathbf{E}'$ and the disturbed magnetic field $\mathbf{B}'$ are induced in the liquid and coupled with $(\mathbf{u}, p)$ through the Navier–Stokes and Maxwell equations [3, 4]. For a liquid with uniform magnetic permeability μ_m, conductivity σ, density ρ_f and

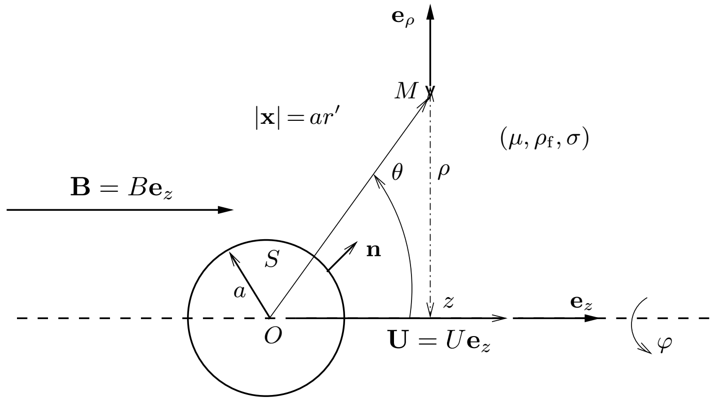

Fig. 1. A highly-slipping sphere S translating in a conducting Newtonian liquid at the velocity $\mathbf{U} = U\mathbf{e}_z$ parallel to a uniform magnetic field $\mathbf{B} = B\mathbf{e}_z$.

viscosity μ, one introduces the magnetic Reynolds number $\mathrm{Rm} = \mu_{\mathrm{m}}\sigma|U|a$ and the Reynolds number $\mathrm{Re} = \rho_{\mathrm{f}}|U|a/\mu$. Here, we assume $\mathrm{Re} \ll 1$, i.e. consider a 'slow viscous' flow. For applications $\mathrm{Rm} \ll \mathrm{Re}$. Accordingly (see, for instance, [5]), the magnetic field is not disturbed by the sphere, i.e. $\mathbf{B}' = \mathbf{B}$. Because $\mathbf{B}$ is uniform and $(\mathbf{u}, p)$ is axisymmetric with no swirl, note that $\mathbf{E}' = \mathbf{0}$ [3, 6]. The quasi-steady MHD creeping flow $(\mathbf{u}, p)$ about the translating sphere here solely depends on the so-called Hartmann number $\mathrm{Ha} = a/d$, where $d = (\sqrt{\mu/\sigma})/B$ denotes the Hartmann layer thickness [7]. In this framework, results for a translating no-slip solid sphere have been obtained asymptotically for $\mathrm{Ha} \ll 1$ and $\mathrm{Ha} \gg 1$ in [8, 9] and numerically for arbitrary $\mathrm{Ha} > 0$ in [10] using a boundary element method. This work derives a new and efficient approach to deal with the case of a translating and highly-slipping sphere whatever $\mathrm{Ha} > 0$.

1. Addressed axisymmetric problem and adopted treatment.

This section presents the considered axisymmetric MHD viscous flow problem and briefly introduces the procedure used to efficiently solve it.

1.1. Governing equations and drag coefficient. The quasi-steady viscous MHD flow $(\mathbf{u}, p)$ about the sphere is subject to a Lorentz body force $\mathbf{j} \wedge \mathbf{B}$, in which the current density $\mathbf{j}$ reads (in the absence of an electric field; see the introduction) $\mathbf{j} = \sigma(\mathbf{u} \wedge \mathbf{B})$. Thus,

$$\mu\nabla^2\mathbf{u} = \nabla p - \sigma B^2(\mathbf{u} \wedge \mathbf{e}_z) \wedge \mathbf{e}_z \text{ and } \nabla \cdot \mathbf{u} = 0 \text{ in D }, \tag{1}$$

where $\mathscr{D}$ denotes the unbounded liquid domain and $\mathbf{x} = \mathbf{OM}$ with O being the sphere center. The flow has a stress tensor $\boldsymbol{\sigma}$ and thus exerts on the sphere surface S the stress $\mathbf{f} = \boldsymbol{\sigma} \cdot \mathbf{n}$, with $\mathbf{n}$ being the unit normal pointing into the liquid. In this paper we consider a highly-slipping sphere. Using on its surface the usual Navier slip condition [11] with a slip length much more larger than the sphere radius yields the boundary condition $\mathbf{f} \wedge \mathbf{n} = \mathbf{0}$ at the sphere boundary. Considering a quiescent liquid far from the sphere, the imposed far-field behaviour and impermeability and zero tangent stress conditions at the sphere boundary are

$$(\mathbf{u}, p) \to (\mathbf{0}, 0) as |\mathbf{x}| \to \infty, \mathbf{u} \cdot \mathbf{n} = U\mathbf{e}_z \cdot \mathbf{n} \text{ and } \mathbf{f} \wedge \mathbf{n} = \mathbf{0} \text{ on } S. \tag{2}$$

The MHD flow $(\mathbf{u}, p)$ about the sphere, the solution to Eqs. (1)–(2), solely depends on the Hartmann number $\mathrm{Ha} = a/d$, with $d = (\sqrt{\mu/\sigma})/B$ being the Hartmann layer tickness [7]. Clearly, $(\mathbf{u}, p)$ is axisymmetric about the $(O, \mathbf{e}_z)$ axis and has no swirl. Accordingly, it applies on the sphere no torque about its center O and a force $\mathbf{F}$ parallel to $\mathbf{e}_z$. The resulting so-called sphere drag coefficient C_{d} is defined as

$$\mathbf{F} = \int_S \mathbf{f} \mathrm{d}S = -4\pi\mu a C_{\mathrm{d}} U \mathbf{e}_z,\ C_{\mathrm{d}} = C_{\mathrm{d}}(\mathrm{Ha}). \tag{3}$$

Note that $C_{\mathrm{d}} = 1$ (see [12]) for $\mathrm{Ha} = 0$, i.e. for the Stokes flow case obtained in the absence of ambient magnetic field and/or for a non-conducting liquid ($\sigma = 0$).

1.2. Advocated treatment. Let use normalized spherical coordinates (r', θ, φ) with $r' = |\mathbf{x}|/a$, polar angle θ in $[0, \pi]$ and azimuthal angle φ in $[0, 2\pi]$. Note that φ is the angle about the $(O, \mathbf{e}_z)$ axis, whereas $z' = z/a = r' \cos\theta$. The normalized flow $(\mathbf{u}', p')$, defined by $\mathbf{u}' = \mathbf{u}/U = u'_r(r', \theta)\mathbf{e}_r + u'_\theta(r', \theta)\mathbf{e}_\theta$ and $p' = ap/(\mu U)$, exerts on the $r' = 1$ sphere surface a normalized stress $\mathbf{f}' = a\mathbf{f}/(\mu U) = f'_r(1, \theta)\mathbf{e}_r + f'_\theta(1, \theta)\mathbf{e}_\theta$. Under these notations, the boundary conditions (2) on the sphere surface become

$$u'_r(1, \theta) = \cos\theta \text{ and } f'_\theta(1, \theta) = 0 \text{ for } \theta \text{ in } [0, \pi]. \tag{4}$$

Referring to [1, 2], Eqs. (1) and the far-field behaviour (2) imply that $(\mathbf{u}', p')$ can be expanded in terms of some fundamental solutions. Taking $l = 1, 2$ and setting $\beta = \text{Ha}$, one actually gets

$$\mathbf{u}' = \mathrm{e}^{\beta z'}\nabla\phi_1 + \mathrm{e}^{-\beta z'}\nabla\phi_2, \ \frac{p'}{\beta} = \mathrm{e}^{\beta z'}\frac{\partial\phi_1}{\partial z'} - \mathrm{e}^{-\beta z'}\frac{\partial\phi_2}{\partial z'}, \ \Delta\phi_l = (-1)^{l+1}\beta\frac{\partial\phi_l}{\partial z'}. \tag{5}$$

Moreover, the above functions ϕ_1 and ϕ_2, which make the flow $(\mathbf{u}', p')$ satisfy the required far-field behaviour (2), take the followings forms

$$\phi_1 = \mathrm{e}^{-\beta z'/2}\sum_{n\geq 0} A_n F_n(\beta r')P_n(\cos\theta), \phi_2 = \mathrm{e}^{\beta z'/2}\sum_{n\geq 0} B_n F_n(\beta r')P_n(\cos\theta), \tag{6}$$

where P_n denotes the usual Legendre polynomial and F_n is related to the usual modified Bessel function $K_{n+1/2}$. More precisely,

$$F_n(t) = t^{-1/2}K_{n+1/2}(t/2), F_n(t) = (\pi)^{1/2}\mathrm{e}^{-t/2}\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!t^{k+1}}. \tag{7}$$

Using Eqs. (5)-(6) gives the quantities $u_r'(1,\theta)$ and $f_\theta'(1,\theta)$ involved by the boundary conditions (4). If F_n' designates the derivative of F_n, one easily arrives at

$$u_r'(1,\theta) = \beta\sum_{n\geq 0}\big\{[A_n\mathrm{e}^{\beta\cos\theta/2} + B_n\mathrm{e}^{-\beta\cos\theta/2}]F_n'(\beta) \\ - \frac{\cos\theta F_n(\beta)}{2}[A_n\mathrm{e}^{\beta\cos\theta/2} - B_n\mathrm{e}^{-\beta\cos\theta/2}]\big\}P_n(\cos\theta), \tag{8}$$

$$f_\theta'(1,\theta) = \sin\theta\sum_{n\geq 0}[A_n\mathrm{e}^{\beta\cos\theta/2} + B_n\mathrm{e}^{-\beta\cos\theta/2}] \\ \times\big\{(\frac{\beta^2}{2})F_n(\beta)\cos\theta P_n(\cos\theta) + 2[F_n(\beta) - \beta F_n'(\beta)]P_n'(\cos\theta)\big\}. \tag{9}$$

As can be easily shown, the MHD flow symmetries imply that $B_n = (-1)^{n+1}A_n$. Thus, one solely needs to determine the coefficients A_n. Let first introduce for $l = 0, 1$ and positive integers n and m the following integrals

$$I^\alpha_{l,n,m} = \int_{-1}^{1}\mathrm{e}^{(\text{Ha}/2)t}t^l P_n(t)P_m(t)\mathrm{d}t, K^\alpha_{n,m} = \int_{-1}^{1}\mathrm{e}^{(\text{Ha}/2)t}P_n'(t)P_m(t)\mathrm{d}t. \tag{10}$$

Taking $m = 1, 3, ..., N{-}1$ (with N being a positive and even integer) and integrating the quantity $u_r'(1,\theta)P_m(\cos\theta)\sin\theta$ over $[0,\pi]$ using Eq. (8) and the boundary condition for $u_r'(1,\theta)$ then first gives

$$\sum_{n=0}^{N}\text{Ha}\big\{2F_n'(\text{Ha})I^\alpha_{0,n,m} - F_n(\text{Ha})I^\alpha_{1,n,m}\big\}A_n = \frac{2\delta_{m1}}{3} \quad \text{for } m = 1, 3, ..., N{-}1 \ , \tag{11}$$

with δ_{nm} being the usual Kronecker symbol. Taking $m = 0, 2, ..., N$ and integrating over $[0,\pi]$ the product $f_\theta'(1,\theta)P_m(\cos\theta)$ using Eq. (9) and the boundary condition $f_{1,\theta}' = 0$ this time yields

$$\sum_{n=0}^{N}\big\{(\text{Ha})^2F_n(\text{Ha})I^\alpha_{1,n,m} - 4\text{Ha}F_n'(\text{Ha})K^\alpha_{n,m} \\ + 4F_n(\text{Ha})K^\alpha_{n,m}\big\}A_n = 0 \text{ for } m = 0, 2, ..., N. \tag{12}$$

Table 1. Computed values of the sphere drag coefficient $C_{\rm d}$ for a few values of $\mathrm{Ha} \leq 20$ and different truncating (even) numbers N.)

N	Ha = 0.1	Ha = 1	Ha = 5	Ha = 10	Ha = 20
4	1.025291	1.277529	2.741242	4.888646	9.847587
10	1.025291	1.277529	2.741354	4.880310	9.439040
20	1.025291	1.277529	2.741354	4.880310	9.439050
40	1.025291	1.277529	2.741354	4.880310	9.439050

The truncated linear system (11)–(12) is first numerically solved to get the coefficients A_n. As a second step, the drag coefficient $C_{\rm d}$ defined by eq. (3) is obtained as $C_{\rm d} = -2\sqrt{\pi}\sum_{n=0}^{N} A_n$ (the proof is similar to the one proposed in [1]).

2. Numerical implementation and results.

This section addresses some numerical and convergence issues and also examines the sphere drag coefficient $C_{\rm d}$ and flow velocity sensitivity to the Hartmann number Ha.

2.1. Some numerical issues. For the given (even) integer N one needs to accurately compute $M = (N+1)(2N+3)$ integrals $I^{\alpha}_{0,n,m}, I^{\alpha}_{1,n,m}$ and $K^{\alpha}_{n,m}$. As N increases, the integer M behaves as $2N^2$ and thus becomes large. Some key induction properties of the Legendre polynomials fortunately make it possible to reduce the task to the computation of only $2(N+1)$ integrals: the integrals $I^{\alpha}_{0,n,n}$ and $J^{\alpha}_{n,n}$ for $n = 0, ..., N$.

The convergence of the sphere drag coefficient $C_{\rm d}$ versus N is illustrated in Table 1 for a few values of the Hartmann number Ha. Clearly, $C_{\rm d}$ quickly converges as N increases and taking $N = 20$ is quite sufficient to reach a nice 6-digit accuracy for $\mathrm{Ha} \leq 20$.

2.2. Drag coefficient sensitivity to the Hartmann number. The sphere drag coefficient $C_{\rm d}$ is plotted versus $\mathrm{Ha} \leq 20$ in Fig. 2 taking $N = 20$.

Clearly, $C_{\rm d}$ increases with and is very sensitive to the Hartmann number Ha. For instance (as seen in Table 1), at Ha = 5 the drag value is about 2.75 times its value for the pure Ha = 0 Stokes flow.

For $\mathrm{Ha} \ll 1$, it is possible to asymptotically solve the linear system (11)–(12). After elementary but lengthy algebra (too long to be reproduced here) one then arrives at the

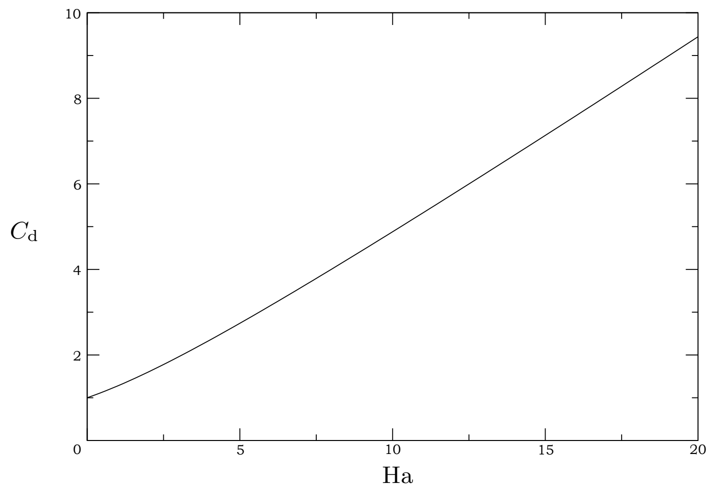

Fig. 2. Sphere drag coefficient $C_{\rm d}$ versus Ha

Table 2. Comparisons between the drag coefficient $C_{\rm d}$ (here computing taking $N = 20$) and its polynomial asymptotic estimate $C_{\rm d}^{\rm a}$ (see Eq. (13)).)

Ha	0.01	0.1	0.3	0.5	0.7	1
$C_{\rm d}$	1.00250	1.02529	1.07759	1.13121	1.18877	1.27753
$C_{\rm d}^{\rm a}$	1.00250	1.02529	1.07759	1.13121	1.18893	1.27813

following handy polynomial formula

$$C_{\rm d} \sim C_{\rm d}^{\rm a} = 1 + {\rm Ha}/4 + 7{\rm Ha}^2/240 - {\rm Ha}^3/960, \quad \text{for Ha being small enough.} \tag{13}$$

As shown in Table 2, our computations are in excellent agreement with Eq. (13) for ${\rm Ha} \leq O(1)$.

2.3. Velocity sensitivity to the Hartmann number. As earlier noted, the flow (normalized) velocity is $\mathbf{u}' = u'_r \mathbf{e}_r + u'_\theta \mathbf{e}_\theta$ with components $u'_r(r', \theta)$ and $u'_\theta(r', \theta)$ calculated from Eqs. (5)–(6) in terms of ${\rm Ha}, (r', \theta)$ and the coefficients A_n and $B_n = (-1)^{n+1} A_n$.

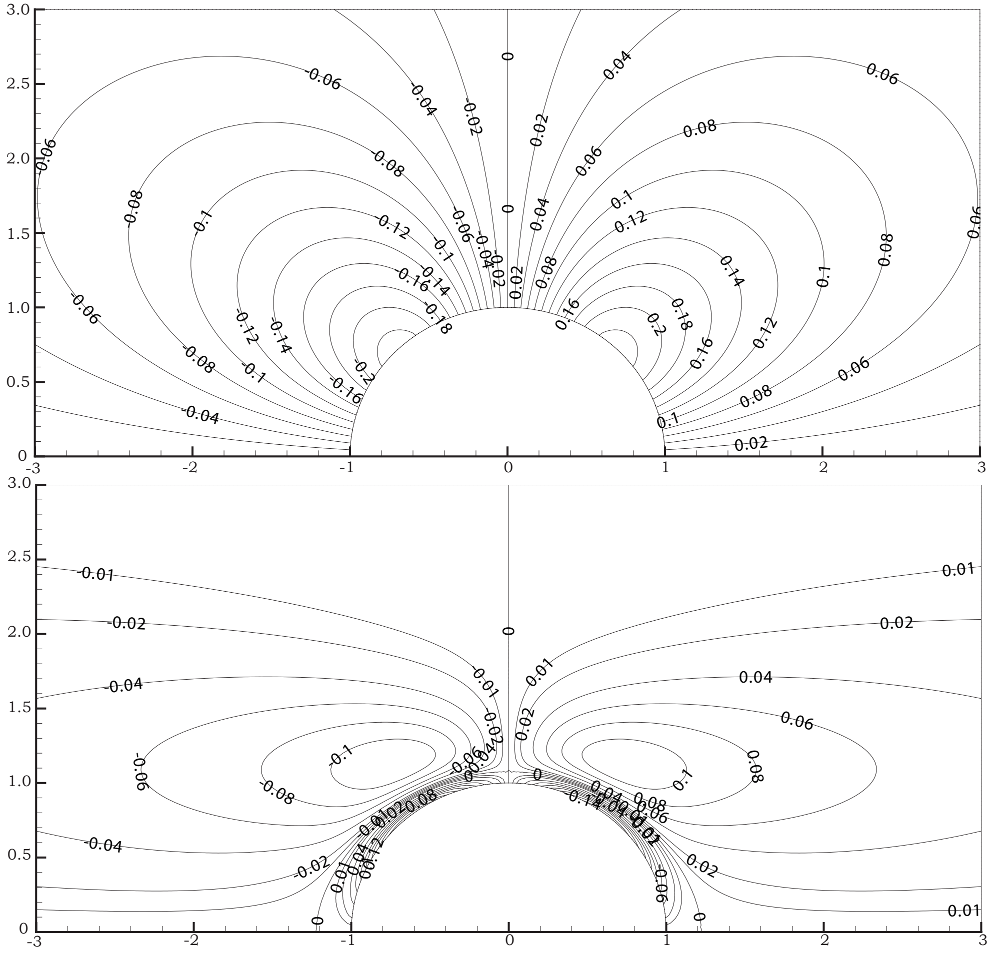

Fig. 3. Isolevel lines in the (z', ρ') half-plane of the normalized velocity component u'_ρ for ${\rm Ha} = 1$ (top) and ${\rm Ha} = 10$ (bottom).

One can also resort to the cylindrical coordinates (ρ, z, φ) with ρ as the distance to the $(O, \mathbf{e}_z)$ axis and $\mathbf{e}_\rho = [\mathbf{x} - z\mathbf{e}_z]/\rho$ as the usual unit local vector (see Fig. 1). Then, $\mathbf{u}'$ rewrites as $\mathbf{u}' = u'_\rho(\rho', z')\mathbf{e}_\rho + u'_z(\rho', z')\mathbf{e}_z$ with

$$u'_\rho = u'_r \sin\theta + u'_\theta \cos\theta,\ u'_z = u'_r \cos\theta - u'_\theta \sin\theta,\ \rho' = r' \sin\theta,\ z' = r' \cos\theta. \qquad (14)$$

The isolevel curves of the velocity components u'_ρ and u'_z have been computed for Ha $= 1, 10$ taking $N = 40$.

The results for u'_ρ are shown in Fig. 3. As expected, $u'_\rho(\rho', -z') = -u'_\rho(\rho', -z')$. Moreover, $|u'_\rho|$ is small (i.e. here at most of order 0.1) in the entire liquid domain and quickly decays in all directions away from the sphere center O for each considered value of Ha. The maximum of $|u'_\rho|$ decreases as Ha increases. Moreover, it is obtained at two symmetric points (replace z' with $-z'$) on the sphere surface when Ha is sufficiently small (here for Ha $= 1$), whereas it is reached at two symmetric points located inside the liquid domain for sufficiently large Ha (here for Ha $= 10$).

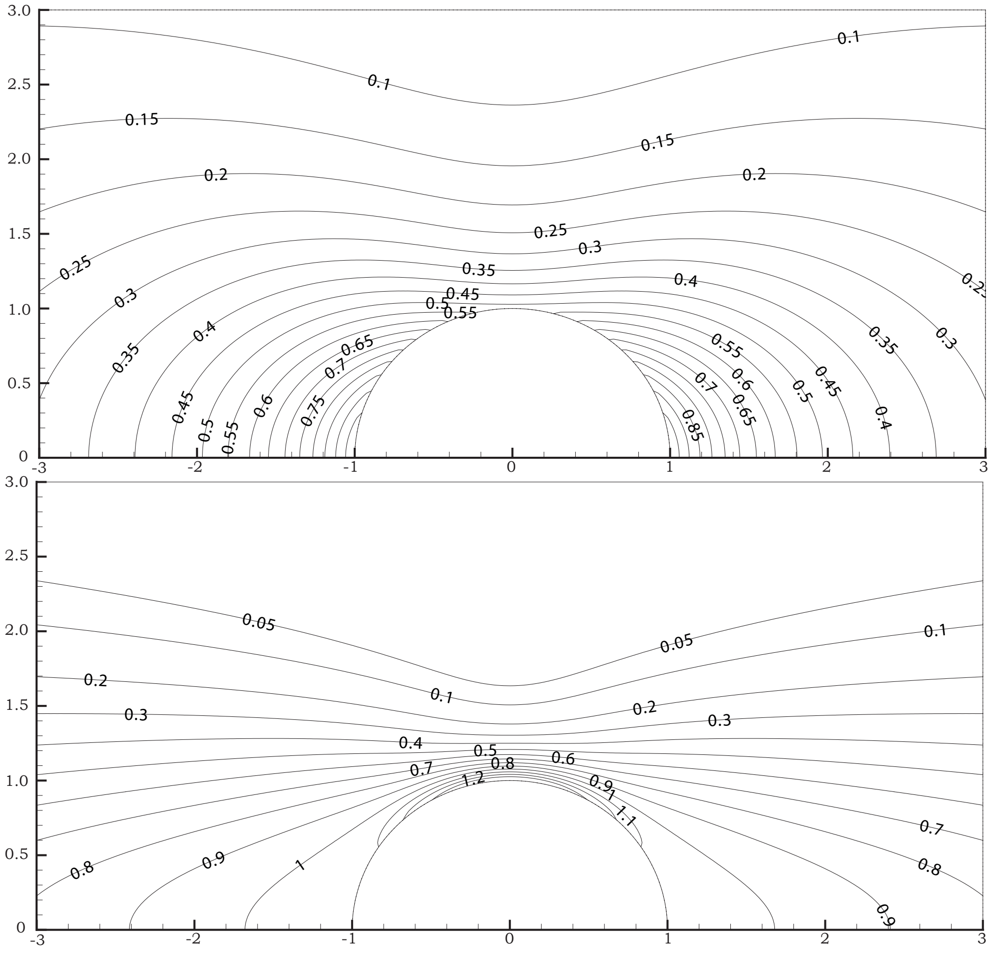

Fig. 4. Isolevel lines in the (z', ρ') half-plane of the normalized velocity component u'_z for Ha $= 1$ (top) and Ha $= 10$ (bottom).

The results for u'_z are displayed in Fig. 4. Here, $u'_z(\rho', -z') = u'_z(\rho', z')$ and $u'_z > 0$ quickly decays away from the $(O, \mathbf{e}_z)$ axis for a given value of z'. Near this $\rho' = 0$ axis u'_z slowly decays in the $z' < 0$ upstream direction and in the $z' > 0$ downstream direction. Moreover, the liquid domain, where $u'_z \geq 0.5$, extends both upstream and downstream as Ha increases. From the impermeability condition (4), $u'_z = 1$ at the two points of the sphere surface located on the axis ($\rho' = 0$ and $|z'| = 1$). For a sufficiently small Hartmann number (here Ha = 1), it appears that $0 < u'_z < 1$ elsewhere in the liquid. In contrast, for Ha = 10 one gets $u'_z > 1$ near the sphere boundary. However, the asymptotic results obtained in [9] when Ha $\gg 1$ for a no-slip solid body or revolution translating parallel to both its axis and a uniform ambient magnetic field can be extended (not done here for conciseness) for a higly-slipping sphere. Therefore, for Ha $\gg 1$ one actually expects the considered normalized axial velocity u'_z to tend to unity in the liquid domain contained within the $\rho' = 1$ cylindrical tube. Note that one can also deduce that $C_\mathrm{d} \sim$ Ha/2 for Ha $\gg 1$.

3. Conclusions.

A new method has been presented and implemented to efficiently calculate the axisymmetric MHD flow about a higly-slipping sphere translating in a conducting liquid parallel to a uniform imposed ambient magnetic field. The procedure provides numerical results both for the sphere drag coefficient C_d and for the flow velocity and pressure about the translating sphere. Both C_d and the flow are found to deeply depend on the problem Hartmann number. Moreover, a simple polynomial handy formula has been derived for C_d assuming a small value of Ha. This formula nicely approximates the numerical results for Ha $\leq O(1)$.

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